ELSEVIER

Journal of Pure and Applied Algebra 125 (1998) 287–303

JOURNAL OF PURE AND APPLIED ALGEBRA

The conditions $Int(R) \subseteq R_S[X]$ and $Int(R_S) = Int(R)_S$ for integer-valued polynomials

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Communicated by M.-F. Roy; received 20 July 1995; revised 19 March 1996

Abstract

Let R be an integral domain with quotient field K and let $Int(R) = \{f \in K[X] \mid f(R) \subseteq R\}$. In this note we determine when $Int(R) = R[X]$ for an arbitrary integral domain R. More generally we determine when $Int(R) \subseteq R_S[X]$ for a multiplicative subset S of R. In the case that R is an almost Dedekind domain with finite residue fields we also determine when $Int(R_S) = Int(R)_S$ for each multiplicative subset S of R, and show that if this holds then finitely generated ideals of $Int(R)$ can be generated by two elements. © 1998 Elsevier Science B.V.

AMS classification: 13F20; 13F05; 13B24

0. Introduction

Let R be an integral domain with quotient field K and let $Int(R)$ be the ring of integer-valued polynomials on R. Thus, $Int(R) = \{f \in K[X] \mid f(R) \subseteq R\}$. The ring $Int(R)$ has been much studied since it was considered in the 1919 articles of Ostrowski [19] and Polya [20] for the case that R is the ring of integers in an algebraic number field. Among the first questions which arise in studying $Int(R)$ is to determine when $Int(R) = R[X]$. This question was considered [5], and was answered for R Noetherian in [22]. This result for the Noetherian case was later used for example in [11] and in [12]. In this note we determine when $Int(R) = R[X]$ for an arbitrary integral domain R. More generally, we determine when $Int(R) \subseteq R_S[X]$ for a multiplicative subset S of R. We also determine when $Int(R_S) = Int(R)_S$ for each multiplicative subset S of R in the case that R is an almost Dedekind domain with finite residue fields. It turns out that these properties put rather strong finiteness conditions on R, and focus attention on a special class of almost Dedekind domains which was defined in [17]. We also show that if R is an almost Dedekind domain with finite residue fields and $Int(R_S) = Int(R)_S$

0022-4049/98/$19.00
PII S0022-4049(96)00107-7

for each multiplicative subset S of R, then finitely generated ideals of $Int(R)$ can be generated by two elements, as in the case that R is Dedekind with finite residue fields.

A question of Brizolis [4, p. 1075] that has been considered by several authors is to determine when $Int(R)$ is a Prüfer domain. In the case of a *Noetherian* ring R it is known that $Int(R_S) = Int(R)_S$ for each multiplicative subset S of R [5], and thus for R Noetherian this question reduces to the case that R is local. Using this and results in [6] on the case that R is a Noetherian valuation ring, it was shown by Chabert [7, Corollaire 6.5] and McQuillan [18] that if R is Noetherian, $Int(R)$ is Prüfer if and only if R is a Dedekind domain with finite residue fields. In [7] Chabert also observed that if R is a domain such that $Int(R)$ is Prüfer, then R is almost Dedekind with finite residue fields. Recall that a domain R is an *almost Dedekind domain* if R_P is a Noetherian valuation ring for each maximal ideal P of R [9, Section 36].

In exploring the above question of Brizolis, various authors have constructed examples of almost Dedekind domains R with finite residue fields for which $Int(R)$ is Prüfer, and examples of such almost Dedekind domains for which $Int(R)$ is not Prüfer. For example, see [8, 10, 16, 17]. In the examples in [10] where $Int(R)$ is not Prüfer, the technique used for showing this is to show that $Int(R) \subseteq R_P[X]$ for some maximal ideal P of R. Indeed condition (1) $Int(R) \nsubseteq R_P[X]$ for each maximal ideal P of R, is easily seen to be necessary for $Int(R)$ to be Prüfer. This brings up the question of characterizing condition (1). This is done in section one of this note. On the other hand, condition (2) $Int(R_P) = Int(R)_{R-P}$ for each maximal ideal P of R, on an almost Dedekind domain R with finite residue fields is obviously sufficient for $Int(R)$ to be Prüfer. This condition is characterized in Section 2. An example in [8] shows that (1) $\not\Rightarrow$ (2) for a general almost Dedekind domain with finite residue fields. In Section 3 we relate the almost Dedekind domains which satisfy (2) to the so-called glad domains defined in [17], and give some results on these classes of almost Dedekind domains. In Section 4 we show that the property that finitely generated ideals of $Int(R)$ are generated by two elements, and some related properties, extend from the case that R is a Dedekind domain with finite residue fields, to the more general case that R satisfies (2).

1. When $Int(R) \subseteq R_S[X]$

In this section we determine when $Int(R) \subseteq R_S[X]$ for an integral domain R and a multiplicative subset S of R. By a *valuation ring*, we mean a ring, possibly with zero-divisors, in which the set of ideals is totally ordered under inclusion. If R is a ring define $d_n(X) = d_n(X_0, X_1, \ldots, X_n) \in R[X_0, X_1, \ldots, X_n]$ by

$$d_n(X) = \prod_{0 \le i < j \le n} (X_j - X_i).$$

We can consider d_n as a function $R^{n+1} \to R$ in the usual way. If $x \in R$ is nilpotent we define *the nilpotence degree of* x as the greatest integer n such that $x^n \neq 0$, and denote it $nd(x)$. If $s \in R$ and I is an ideal of R let $D(s)$ denote $\{P \in \mathrm{Spec}(R) \mid s \notin P\}$ and $V(I) = \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$. We denote the non-negative integers by $\mathbb{Z}_+$, and the cardinality of a set A by $|A|$. If $f \in B[X]$ for some ring B containing R as a subring, we denote the R-submodule of B generated by the coefficients of f by $c_R(f)$, or by $c(f)$ if the reference to R is clear. The following lemma shows the relevance of the polynomials $d_n(X)$.

Lemma 1.1. *Let R be an integral domain with quotient field K. If $f = g/a \in Int(R)$, $g \in R[X]$, $a \in R$, then $d_n(R^{n+1}) \subseteq (aR :_R c(g))$.*

Proof. Write $g = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_0 \in R[X]$, $a_i \in R$. Let $x_0, x_1, \ldots, x_n \in R$ and let

$$A = \begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{pmatrix}, \quad B = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad C = \begin{pmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_n) \end{pmatrix}.$$

Since the entries of C are in aR by hypothesis, then multiplying both sides of the equation $AB = C$ by the adjoint of A we see that $da_i \in aR$ for each i, with $d = d_n(x_0, \ldots, x_n) = \prod_{0 \le i < j \le n}(x_j - x_i)$. Thus, we have $d_n(R^{n+1}) \subseteq (aR :_R c(g))$. $\square$

If I is an ideal of R and $A = R/I$, then it is clear that $d_n(R^{n+1}) \subseteq I$ if and only if $d_n(A^{n+1}) = \{0\}$. In the next few lemmas we give some consequences of $d_n(R^{n+1}) = \{0\}$. Recall that a ring R is *arithmetical* if R_P is a valuation ring for each maximal ideal P of R.

Lemma 1.2. *Let R be a ring. If $d_n(R^{n+1}) = 0$ for a positive integer n then the following hold:*

(a) *The set $\{|R/P| \mid P \in \mathrm{Spec}(R)\}$ is bounded.*

(b) *The set $\{nd(x) \mid x \in nil(R)\}$ is bounded.*

If R is arithmetical (b) *can be strengthened to*

(c) *$J^k = 0$ for some k where J is the Jacobson radical of R.*

Proof. For the proof of (a) we claim that $n \ge |R/P|$ for each $P \in \mathrm{Spec}(R)$. Indeed if some $x_0, x_1, \ldots, x_n \in R$ have distinct images in R/P for some $P \in \mathrm{Spec}(R)$, then we have $\prod_{0 \le i < j \le n}(x_j - x_i) = d_n(x_0, \ldots, x_n) \notin P$, a contradiction.

For the proof of (b) let $x \in \mathrm{nil}(R)$. Since $d_n(1, x, x^2, \ldots, x^n) = 0$, then writing each factor $(x^j - x^i)$ of $d_n(1, x, x^2, \ldots, x^n)$ as $x^i(x^{j-i} - 1)$ and observing that $(x^{j-i} - 1)$ is a unit in R, we see that for some $t > 0$, $x^t = 0$ for all $x \in \mathrm{nil}(R)$.

To show that (b) $\Rightarrow$ (c) if R is arithmetical let t be such that $x^t = 0$ for all $x \in \mathrm{rad}(R)$. It suffices to show that $x_1x_2 \cdots x_t = 0$ for $x_1, x_2, \cdots, x_t \in \mathrm{rad}(R)$. For this it suffices to show $x_1x_2 \cdots x_t = 0$ in R_P for each prime ideal P of R. But the result is clear in this case since R_P is a valuation ring, and hence $(x_1, x_2, \ldots, x_t)R_P$ is principal. $\square$

Corollary 1.3. *If R is a ring with Jacobson radical J, the conditions* (a) *and* (c) *of Lemma* 1.2 *imply*:

(d) *The group of units R^* of R has finite exponent.*

Proof. It follows from condition (a) that $(R/J)^*$ has finite exponent. Suppose we have shown that $(R/J^{i-1})^*$ has finite exponent. We have the exact sequence

$$0 \to 1 + (J^{i-1}/J^i) \to (R/J^i)^* \to (R/J^{i-1})^* .$$

Also, the multiplicative group $1 + J^{i-1}/J^i$ is isomorphic to the additive group J^{i-1}/J^i, since $(1+x)(1+y) = 1+x+y$ for $x, y \in J^{i-1}/J^i$. It follows from conditions (a) and (c) that $kJ^{i-1} \subseteq kR = 0$ for some positive integer k, and thus $1 + J^{i-1}/J^i$ has finite exponent. Therefore $(R/J^i)^*$ does also. $\square$

Before giving our main result on when $Int(R) \subseteq R_S[X]$ for a multiplicative subset S of R, we remind the reader that a difficulty with $Int(R)$ in general is that $Int(R_S) = Int(R)_S$ can fail for a multiplicative subset S of R. On the positive side there is the following result of P.-J. Cahen and J.-L. Chabert.

Proposition 1.4 (Cahen and Chabert [5, p. 303, Corollaries 4, 5]). *If S is a multiplicative subset of R then*

(1) *$Int(R)_S \subseteq Int(R_S)$.*

(2) *If R is Noetherian, then $Int(R)_S = Int(R_S)$.*

Proof. We give an alternate proof for (1) that is shorter than the one in [5]. For this it suffices to show that $Int(R) \subseteq Int(R_S)$. Let $f \in Int(R)$ have degree n and assume $Int(R_S)$ contains the members of $Int(R)$ of degree $< n$. Let $r/s \in R_S$, $r \in R$, $s \in S$ and consider $s^n f(X) = (s^n f(X) - f(sX)) + f(sX)$. Since $s^n f(X) - f(sX) = g_s(X) \in Int(R)$ and has degree $< n$, $g_s(X) \in Int(R_S)$. Then $s^n(f(\frac{r}{s})) = g_s(\frac{r}{s}) + f(s(\frac{r}{s})) \in R_S$, and hence $f(\frac{r}{s}) \in R_S$.

A short and simple proof of (2) is given in [22, Proposition 4]. $\square$

Theorem 1.5. *Let R be an integral domain with quotient field K and let S be a multiplicative subset of R. The following are equivalent*:

(1) *$Int(R) \not\subseteq R_S[X]$.*

(2) *There exists a positive integer n, and elements $a, b \in R$ such that $(aR :_R b)R_S \neq R_S$ and $d_n(R^{n+1}) \subseteq (aR :_R b)$.*

(3) *There exist $a, b \in R$ with $(aR :_R b)R_S \neq R_S$ such that the two sets $\{|R/P| \mid P \in V(aR :_R b)\}$ and $\{nd(x) \mid x \in nil(R/(aR :_R b))\}$ are finite.*

Proof. (1) $\Rightarrow$ (2) Let f be an element of $Int(R) - R_S[X]$ of minimal degree. Write $f = g/a$, with $g = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \in R[X]$, a, $a_i \in R$. Then $a_n \notin aR_S$.

Indeed if $a_n = ra/s$ for some $r \in R$, $s \in S$, we would have $sf = sg/a \in Int(R) - R_S[X]$, and hence $(sa_{n-1}X^{n-1} + sa_{n-2}X^{n-2} + \cdots + sa_0)/a \in Int(R) - R_S[X]$. By Lemma 1.1 we have $d_n(R^{n+1}) \subseteq (aR :_R a_n)$.

(2) $\Rightarrow$ (3) This follows from Lemma 1.2.

(3) $\Rightarrow$ (1) Let $\{q_1, \ldots, q_n\} = \{|R/P| \mid (aR :_R b) \subseteq P\}$. The monic polynomial $f(X) = \prod_{i=1}^n (X^{q_i} - X)$ satisfies $f(R) \subseteq \mathrm{rad}(aR :_R b)$, and $g(X) = f(X)^j$ satisfies $g(R) \subseteq (aR :_R b)$ for some j. Then we have $bg(X)/a \in Int(R)$, and since the leading coefficient of $bg(X)$, which is b, is not contained in aR_S, $bg(X)/a \notin R_S[X]$. □

Corollary 1.6. *If R is a Prüfer domain and S a multiplicative subset of R, the conditions in Theorem* 1.5 *are equivalent to*:

(4) *There exists $a, b \in R$ with $(aR :_R b)R_S \neq R_S$ such that the set $\{|R/P| \mid P \in V(aR :_R b)\}$ is finite and $(aR :_R b)$ contains a power of its radical.*

Proof. This follows from Lemma 1.2. □

Corollary 1.7. *Let R be an integral domain with quotient field K. The following are equivalent*:

(1) *$Int(R) \neq R[X]$.*

(2) *There exists a positive integer n, and elements $a, b \in R$ such that $b \notin aR$ and $d_n(R^{n+1}) \subseteq (aR :_R b)$.*

(3) *There exist $a, b \in R$ with $b \notin aR$ such that the two sets $\{|R/P| \mid P \in V(aR :_R b)\}$ and $\{nd(x) \mid x \in nil(R/(aR :_R b)\}$ are bounded.*

Proof. Take $S = \{1\}$ in the above theorem. □

Corollary 1.8 (Shibata et al. [22, Theorem 2]). *If R is a Noetherian domain, $Int(R) \neq R[X]$ if and only if for some prime P of the form $P = (aR :_R b)$, the field R/P is finite.*

Proof. If $Int(R) \neq R[X]$ then $Int(R) \nsubseteq R_P[X]$ for some maximal ideal P of R. Then by Theorem 1.5 there exist $a, b \in R$ with $(aR :_R b)R_P \neq R_P$ such that the two sets $\{|R/Q| \mid Q \in V(aR :_R b)\}$ and $\{nd(x) \mid x \in \mathrm{nil}(R/(aR :_R b))\}$ are finite. In particular, R/P is finite. Also P is a minimal prime of $(aR :_R b)$. That is P is a weak Bourbaki prime of aR. Since R is Noetherian, P is an associated prime of aR; that is $P = (aR :_R c)$ for some $c \in R$. The converse is clear. □

Examples of almost Dedekind domains R having finite residue fields for which the rings $Int(R)$ are not Prüfer are constructed in [10, 8]. Since the results in this section cast some light on these examples we review briefly the method used for constructing them. This construction allows us to show that the condition that $Int(R) \nsubseteq R_M[X]$ for each maximal ideal M of R, is not equivalent to the condition, considered in the next section, that $Int(R)_S = Int(R_S)$ for each multiplicative subset S of R. For the details of this construction see [10], where it is given in much greater generality than needed here. If $A \subseteq B$ are Dedekind domains and $PB = Q_1^{e_1} Q_2^{e_1} \cdots Q_n^{e_n}$ for a prime ideal P of A and primes $Q_1, \dots, Q_n$ of B, we write $e(Q_i, P)$ for the ramification index e_i, and $[B/Q_i : A/P]$ for the residue degree.

Let $P_0, \dots, P_n$ be prime ideals of the ring of integers A_0 of a number field K_0, let $K_0 \subseteq K_1 \subseteq \cdots$ be a sequence of finite extension fields and let $K = \bigcup K_i$. Let $D_0 = (A_0)_S$, where $S = A_0 - (P_1 \cup \cdots \cup P_n)$, and let D_i and D be the integral closures of D_0 in K_i and K, respectively. For each $j \in \{1, 2, \dots, n\}$ consider a sequence of prime ideals Q_i of D_i lying over P_j such that $Q_{i+1} \cap D_i = Q_i$. Then by [1, Corollary 3.6], D is almost Dedekind if and only if

(a′) for each such sequence of prime ideals the set of ramification indices $\{e(Q_i, P_j) \mid i \in \mathbb{Z}_+\}$ is bounded.

For D to have finite residue fields it is necessary and sufficient that

(b′) for each such sequence of prime ideals the set of residue degrees $\{[D_i/Q_i : D_0/P_j] \mid i \in \mathbb{Z}_+\}$ is bounded.

To produce an example of an almost Dedekind domain R with finite residue fields for which $Int(R)$ is not Prüfer, Gilmer shows [10, Example 14] that condition (b) of the following theorem is strictly stronger than condition (b′). He also shows [10, Theorem 13] that (b) is a necessary condition for $Int(R) \nsubseteq R_M[X]$ for some maximal ideal M of R. (The condition $Int(R) \nsubseteq R_M[X]$ is clearly necessary for $Int(R)$ to be Prüfer since every overring of a Prüfer domain is Prüfer.) To furnish more examples where $Int(R) \nsubseteq R_M[X]$ and thus to answer some questions in [10], Chabert shows [8, Example 6.2] that condition (a) of the following theorem is strictly stronger than condition (a′), and [8, Lemma 1.6] that (a) is a necessary condition for $Int(R) \nsubseteq R_M[X]$ for some maximal ideal M of R. In the proof of the following theorem we use Theorem 1.5 to prove that (a) and (b) are necessary for $Int(R) \nsubseteq R_M[X]$. As observed in [10] and [8] this does not require that D_0 be Dedekind. We also give the converse in the important case that D_0 is Dedekind. This part of Theorem 1.9, together with [8, Example 6.5] shows that condition (1) of Theorem 1.9 is strictly weaker than the condition that $Int(R)_S = Int(R_S)$ for each multiplicative subset S of R, for R an almost Dedekind domain with finite residue fields.

Theorem 1.9. *Let D_0 be an almost Dedekind domain with quotient field K_0. Let $K_0 \subseteq K_1 \subseteq \cdots$ be a sequence of finite extension fields with $K = \bigcup K_i$. Let D_i and D be the integral closures of D_0 in K_i and K, respectively. Consider the following statements*:

(1) *$Int(D) \nsubseteq D_M[X]$ for each maximal ideal M of D.*

(2) (a) *For each maximal ideal P_0 of D_0, $\{e(M, P_0) \mid M \in Spec(D)$ with $M \cap D_0 = P_0\}$ is bounded*; *and*

(b) *for each maximal ideal P_0 of D_0, $\{|D/M| \mid M \in Spec(D)$ with $M \cap D_0 = P_0\}$ is bounded.*

Then (1) $\Rightarrow$ (2). *In particular, if $Int(D)$ is Prüfer then* (a) *and* (b) *hold. If D_0 is Dedekind, then* (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2)(b) Assume that (b) fails for the maximal P_0 of D_0. Since D_1 has only finitely many maximal ideals lying over P_0, for one of these, say P_1, the set $\{|D/M| \mid M \in \mathrm{Spec}(D)$ with $M \cap D_1 = P_1\}$ is unbounded. Similarly, since D_2 has only finitely many maximal ideals lying over P_1, for one of these, say P_2, the set $\{|D/M| \mid M \in \mathrm{Spec}(D)$ with $M \cap D_2 = P_2\}$ is unbounded. We similarly get a prime P_i of D_i for each i. Let $M = \bigcup_{i=1}^{\infty} P_i$, a maximal ideal of D.

Since D is Prüfer, the ideals $(aD :_D b)$ are finitely generated. Thus, to show $Int(D) \subseteq D_M[X]$ it suffices by Theorem 1.5 to show that for each finitely generated ideal $I \subseteq M$, the set $\{|D/P| \mid P$ is a maximal ideal of D containing $I\}$ is unbounded. But if $I = (a_1, \ldots, a_n)D$, then $\{a_1, \ldots, a_n\} \subseteq D_i \cap M = P_i$ for some i. Thus, $\{|D/P| \mid P$ is a maximal ideal of D containing $I\} \supseteq \{|D/P| \mid P$ is a maximal ideal of D lying over $P_i\}$ is unbounded.

(1) $\Rightarrow$ (2)(a) Assume that (a) fails for the maximal P_0 of D_0. Since D_1 has only finitely many maximal ideals lying over P_0, for one of these, say P_1, the set $\{e(M, P_1) \mid M$ is a maximal ideal of D lying over $P_1\}$ unbounded. Similarly, since D_2 has only finitely many maximal ideals lying over P_1, for one of these, say P_2, the set $\{e(M, P_2) \mid M$ is a maximal ideal of D lying over $P_2\}$ unbounded. Let $M = \bigcup_{i=1}^{\infty} P_i$, a maximal ideal of D.

To show $Int(D) \subseteq D_M[X]$ it suffices by Theorem 1.5 to show that for each finitely generated ideal $I \subseteq M$, $\{nd(x) \mid x \in \mathrm{nil}(D/I)\}$ is unbounded. But if $I = (a_1, \ldots, a_n)D$, then $\{a_1, \ldots, a_n\} \subseteq D_i \cap M = P_i$ for some i. Let $t_i \in \mathrm{rad}((a_1, \ldots, a_n)D_i)$ generate $P_i(D_i)_{P_i}$. Then in D/I we have $0 \neq \overline{t_0} = \bar{s}\, (\overline{t_i})^{e(P_i, P_0)}$ where $\bar{s}$ is a unit of D_M/ID_M. So the nilpotency degree of D/I is not bounded.

(2) $\Rightarrow$ (1) Assume that D_0 is Dedekind, that (a) and (b) hold, and let M be a maximal ideal of D. To show that $Int(D) \nsubseteq D_M[X]$ it suffices by Theorem 1.5 to find elements $a, b \in D$ with $(aD :_D b)D_M \neq D_M$ such that the two sets of integers $\{|D/P| \mid P \in V(aD :_D b)\}$ and $\{nd(x) \mid x \in \mathrm{nil}(D/(aD :_D b))\}$ are bounded. Let $P_0 = D_0 \cap M$ and let $a \in P_0$ be such that $a(D_0)_{P_0} = P_0(D_0)_{P_0}$. Since P_0 is finitely generated, there exists $s \in D_0 - P_0$ such that $sP_0 \subseteq aD_0$. Then $P_0 \subseteq (aD_0 :_{D_0} s)$, and since P_0 is maximal we have $P_0 = (aD_0 :_{D_0} s)$. Since D_0 is Prüfer, D is a flat D_0-module and hence $P_0 D = (aD_0 :_{D_0} s)D = (aD :_D s)$. Thus by hypothesis (b), the set $\{|D/P| \mid P \in V(aD :_D s)\}$ is bounded. To show that the set $\{nd(x) \mid x \in \mathrm{nil}(D/(aD :_D s))\}$ is bounded, by hypothesis (a) we may choose $k \in \mathbb{Z}_+$ such that $e(P, P_0) < k$ for every $P \in V(P_0 D)$. Let $x \in D$ be such that its image $\bar{x}$ is in $\mathrm{nil}(D/(aD :_D s))$. Since $(aD :_D s) = P_0 D$, we get $x \in P$ for each $P \in V(P_0 D)$. But $x \in D_i$ for some i, and then $x \in P_i$ for each $P_i \in V(P_0 D_i)$. Let $P_0 D_i = Q_1^{e_1} \cap \cdots \cap Q_n^{e_n}$, where the Q_i are the primes of D_i lying over P_0. Then $x \in Q_i$ for each i and hence $x^k \in Q_1^{e_1} \cap \cdots \cap Q_n^{e_n} = P_0 D_i \subseteq P_0 D = (aD :_D s)$. $\square$

2. A characterization of when $Int(R_P) = Int(R)_{R-P}$

In this section we characterize those almost Dedekind domains R such that $Int(R_P) = Int(R)_{R-P}$ for each maximal ideal P of R.

Lemma 2.1. *Let R be an integral domain and let $P \in Spec(R)$ be such that R/P is finite and $Int(R_P) = Int(R)_{(R-P)}$. If $t \in R$ is such that $tR_P = PR_P$ and $\{a_0, \dots, a_{q-1}\}$ is a set of representatives of R/P, then there exists $s \in R - P$ such that for each $Q \in D(s) \cap V(tR)$, we have $tR_Q = QR_Q$ and $\{a_0, \dots, a_{q-1}\}$ is a set of representatives of R/Q.*

Proof. Let $g = (X - a_0)(X - a_1) \cdots (X - a_{q-1})$ and $f = g/t$. Then $f \in Int(R_P)$ which is $Int(R)_{(R-P)}$ by hypothesis. Let $s_1 \in R - P$ be such that $s_1 f \in Int(R)$, let $s_2 = d_{q-1}(a_0, \dots, a_{q-1})$ and let $s = s_1 s_2$.

Let $Q \in D(s) \cap V(tR)$. Since $s_2 \notin Q$, $a_0, \dots, a_{q-1}$ represent distinct cosets in R/Q, and since $sg(x) \in tR \subseteq Q$ for each $x \in R$, $\{a_0, \dots, a_{q-1}\}$ is a complete set of representatives of R/Q. Let $x \in Q$. Then $sg(x + a_0)/t = sf(x + a_0) \in R \Rightarrow sg(x + a_0) \in tR \subseteq Q$. But $sg(x + a_0) = sx(x + a_0 - a_1) \cdots (x + a_0 - a_{q-1})$, and since $s_3 = s(x + a_0 - a_1) \cdots (x + a_0 - a_{q-1}) \notin Q$, $x = (xs_3)/s_3 \in tR_Q$. Therefore, $tR_Q = QR_Q$. $\square$

Lemma 2.2. *Let P and Q be prime ideals of a ring R and let $f : R \to R/P$ and $g : R \to R/Q$ be the canonical maps. The following statements are equivalent.*

(a) *Each set $T \subseteq R$ of representatives for R/P is also a set of representatives for R/Q.*

(b) *There exists a set $T \subseteq R$ which is a set of representatives for both R/P and R/Q.*

(c) *There is a bijection $\varphi : R/P \to R/Q$ such that $\varphi \circ f = g$.*

Proof. The implication (a) $\Rightarrow$ (b) is clear. For (b) $\Rightarrow$ (c) if T is a set of representatives for both R/P and R/Q, it is clear that φ can be defined as $g_1 \circ f_1^{-1}$ where f_1, g_1 are the restrictions of f, g to T. For (c) $\Rightarrow$ (a) let $\varphi : R/P \to R/Q$ be such that $\varphi \circ f = g$. Let T be a set of representatives of R/P. Then $R/Q = \{\varphi(f(a)) \mid a \in T\} = \{g(a) \mid a \in T\}$, and $g(a) \neq g(b)$ for $a \neq b \in T$. $\square$

Theorem 2.3. *Let R be an almost Dedekind domain with finite residue fields and let M be a maximal ideal of R. Then $Int(R_M) = Int(R)_{(R-M)}$ if and only if the following hold for each nonzero $a \in M$:*

(a) *The radical of aR is finitely generated; and*

(b) *there is a partition $\{\mathcal{T}_1, \dots, \mathcal{T}_m\}$ of $V(aR)$ into subsets which are open in $V(aR)$ such that for each $i \in \{1, \dots, m\}$ there exists a set $T_i \subseteq R$ which is a set of representatives of R/Q for each $Q \in \mathcal{T}_i$.*

Proof. ($\Rightarrow$) Let $J = \mathrm{rad}(aR)$. Since R is almost Dedekind, by Lemma 2.1 we get that for each $Q \in V(aR)$ there exists $t(Q) \in J$, a set $T(Q) \subseteq R$ and $s(Q) \in R - Q$ such that for each $P \in D(s(Q)) \cap V(J)$ we have that $t(Q)R_P = PR_P$ and $T(Q)$ is a set of representatives of R/P. Since $\{D(s(Q)) \mid Q \in V(J)\}$ is an open cover of the compact set $V(J)$ there exist $s_1, \dots, s_k \in R$ and corresponding elements $t_1, \dots, t_k \in J$ and finite subsets $T_1, \dots, T_k$ of R such that if $Q \in V(J)$ then for some i we have $Q \in D(s_i)$, T_i

is a set of representatives of R/Q, and $QR_Q = t_iR_Q$. It follows that $J = (a, t_1, \ldots, t_k)R$. This proves (a).

To prove (b), observe that we can define a relation $\sim$ on the set $V(aR)$ by $P \sim Q$ if there is a subset $T \subseteq R$ such that T is a set of representatives for both R/P and R/Q. It is clear that $\sim$ is reflexive and symmetric, and it follows from Lemma 2.2 that $\sim$ is transitive. By the above paragraph there are only finitely many equivalence classes $\mathscr{T}_1, \ldots, \mathscr{T}_m$ of $\sim$. Then it follows from Lemma 2.1 that each $\mathscr{T}_i$ is open in the Zariski topology on $V(aR)$ (and thus also closed).

($\Leftarrow$) It suffices to show that $Int(R_M) \subseteq (Int(R))_{(R-M)}$ by Proposition 1.4. For this it suffices to show that there exists a basis $\{g_n \mid n \geq 0\}$ of $Int(R_M)$ as an R_M-module such that for each n there exists $u_n \in R - M$ with $u_n g_n \in Int(R)$.

Let $a \in M$ be such that $aR_M = MR_M$. Then $\operatorname{rad}(aR) = J$ is finitely generated by hypothesis. Also by hypothesis there is a partition $\{\mathscr{T}_1, \ldots, \mathscr{T}_m\}$ of $V(aR)$ into open subsets $\mathscr{T}_i$ such that for each $i \in \{1, \ldots, m\}$ we have a set $T_i \subseteq R$ which is a set of representatives of R/Q for each $Q \in \mathscr{T}_i$. Assume $M \in \mathscr{T}_1$. Let $\mathscr{T}_1' = \{Q \in \mathscr{T}_1 \mid aR_Q = QR_Q\}$, and let $\mathscr{T}_1'' = \{Q \in \mathscr{T}_1 \mid aR_Q \neq QR_Q\}$. It follows from Lemma 2.1 that $\mathscr{T}_1'$ is open in the Zariski topology on $V(aR)$. We claim that $\mathscr{T}_1''$ is also open. To see this assume $aR_Q = Q^nR_Q$ for some prime $Q \in V(aR)$. There exists $t \in J$ such that $tR_Q = QR_Q$, and then $a/1 = ut^n/v$ for some $u, v \in R - Q$. Then by the case $n = 1$ there exists $s \in R - Q$ such that $tR_H = HR_H$ for each $H \in D(s) \cap V(tR)$. Then $tR_H = HR_H$ for each $H \in D(uvs) \cap V(tR)$. But $D(uvs) \cap V(tR) = D(uvs) \cap V(aR)$, and thus $aR_H = t^nR_H = H^nR_H$ for each $H \in D(uvs) \cap V(aR)$. Since $\mathscr{T}_1'' = \bigcup_{n>1} \mathscr{S}_n$ where $\mathscr{S}_n = \{Q \in \mathscr{T}_1 \mid aR_Q = Q^nR_Q\}$, $\mathscr{T}_1''$ is open.

We now have the partition $\{\mathscr{T}_1', \mathscr{T}_1'', \mathscr{T}_2, \ldots, \mathscr{T}_m\}$ of $V(aR)$ into sets which are open in the Zariski topology on $V(aR)$. We now change notation and denote $\mathscr{T}_1'$ by $\mathscr{T}_0$, $\mathscr{T}_1''$ by $\mathscr{T}_1$ and let $\mathscr{T}_2, \ldots, \mathscr{T}_m$ retain their former meanings. Then for each positive integer k we can write

$$R/a^kR = (R/a^kR)e_{k\,0} \oplus \cdots \oplus (R/a^kR)e_{k\,m},$$

for some idempotents $e_{k\,i}$ of R/a^kR, where the image of $e_{k\,i}$ in $(R/a^kR)_Q$ is the identity if $Q \in \mathscr{T}_i$, and the image of $e_{k\,i}$ in $(R/a^kR)_Q$ is zero if $Q \in V(aR) - \mathscr{T}_i$ [3, Proposition 15, p. 103]. Let $T = \{a_0, \ldots, a_{q-1}\} \subseteq R$ be a set of representatives of R/Q for each $Q \in \mathscr{T}_0 \cup \mathscr{T}_1$. For each $n \in \mathbb{Z}_+$ write n in its q-adic expansion

$$n = u_0 + u_1q + u_2q^2 + \cdots + u_{r_n}q^{r_n}, \quad 0 \leq u_j < q.$$

Define

$$s_n = a_{u_0} + a_{u_1}a + a_{u_2}a^2 + \cdots + a_{u_{r_n}}a^{r_n}.$$

Define $f_0, f_1, \ldots \in K[X]$ by

$$f_0 = 1, \; f_j = X(X - s_1)(X - s_2) \cdots (X - s_{j-1}).$$

Let

$$k_n = \sum_{j=1}^{\infty} \left[\frac{n}{q^j}\right] \quad \text{and} \quad g_n = \frac{f_n}{a^{k_n}}.$$

Since $\{a_0,\dots,a_{q-1}\}$ is a set of representatives of R/Q and $aR_Q = QR_Q$ for each $Q \in \mathscr{T}_0$, $\{g_n \mid n \geq 0\}$ is an R_Q-basis for $Int(R_Q)$ for each $Q \in \mathscr{T}_0$ by [10, Proposition 9]. Fix n and let $d_i \in R$ be a preimage of $e_{k_n i}$ for each $i \in \{1,2,\dots,m\}$. Since $d_0 \equiv 1 \bmod (a^{k_n}R_M)$ then $d_0 \notin M$. Since $d_0 \in a^{k_n}R_Q$ for each $Q \in \mathscr{T}_1 \cup \mathscr{T}_2 \cup \dots \cup \mathscr{T}_m$, then $d_0 f_n$ has the property that $d_0 f_n(R) \subseteq a^{k_n}R_Q$ for each $Q \in \mathscr{T}_1 \cup \mathscr{T}_2 \cup \dots \cup \mathscr{T}_m$. Thus, $d_0 g_n(R) \subseteq R_Q$ for each $Q \in \mathscr{T}_1 \cup \mathscr{T}_2 \cup \dots \cup \mathscr{T}_m$. Also clearly $d_0 g_n(R) \subseteq \cap\{R_Q \mid Q \notin V(aR)\}$ and $d_0 g_n(R) \subseteq \cap\{R_Q \mid Q \in \mathscr{T}_0\}$. Thus $d_0 g_n(R) \subseteq \cap\{R_Q \mid Q \in \operatorname{Spec}(R)\} = R$. Therefore, $d_0 g_n \in Int(R)$. □

Theorem 2.4. *Let R be an almost Dedekind domain with finite residue fields. The following statements are equivalent:*

(1) *$Int(R_P) = Int(R)_{(R-P)}$ for each maximal ideal P of R.*

(2) *$Int(R_S) = Int(R)_S$ for each multiplicative subset S of R.*

(3) *The following hold for each nonzero $a \in R$:*

(a) *The radical of aR is finitely generated; and*

(b) *there is a partition $\{\mathscr{T}_1,\dots,\mathscr{T}_m\}$ of $V(aR)$ into subsets which are open in $V(aR)$ such that for each $i \in \{1,\dots,m\}$ there exists a set $T_i \subseteq R$ which is a set of representatives of R/Q for each $Q \in \mathscr{T}_i$.*

Proof. That (2) implies (1) is clear, and the statements (1) and (3) are equivalent by Theorem 2.3. Thus, it suffices to show that (3) implies (2). Let S be a multiplicative subset of R. We may check the equality $Int(R_S) = Int(R)_S$ locally at primes of R_S. Thus let $P \in \operatorname{Spec}(R)$ with $P \cap S = \emptyset$. Since the properties (a) and (b) clearly pass from R to R_S, from Theorem 2.3 we get $Int(R_S)_{(R_S-PR_S)} = Int((R_S)_{PR_S}) = Int(R_P) = Int(R)_{(R-P)} = (Int(R)_S)_{(R_S-PR_S)}$. □

In the following corollary we collect a few immediate consequences of Theorems 2.3 and 2.4.

Corollary 2.5. *Let R be an almost Dedekind domain and let I be a nonzero finitely generated ideal of R. If R/P is finite and $Int(R_P) = Int(R)_{(R-P)}$ for each $P \in V(I)$, then the following hold:*

(a) *There exists a finite set $T \subseteq R$ which contains a set of representatives of R/Q for each $Q \in V(I)$. In particular, the set $\{|R/P| \mid P \in V(I)\}$ is finite.*

(b) *$rad(I)$ is finitely generated.*

(c) *The group of units $(R/I)^*$ of R/I has finite exponent.*

(e) *If R has Noetherian spectrum, then R is Dedekind.*

Proof. Part (a) follows easily from Theorem 2.3. For the proof of (b) let $I = (t_1, \dots, t_n)$. Then $J_i = \mathrm{rad}(t_i R)$ is finitely generated by Theorem 2.3, and $\mathrm{rad}(I) = J_1 + \cdots + J_n$, as can be seen by checking locally. Thus $\mathrm{rad}(I)$ is finitely generated. The proof of part (c) follows from Corollary 1.3. For (e) recall that R has Noetherian spectrum if and only if each prime ideal of R is the radical of a finitely generated ideal [13, p. 276]. Thus since the radical of a finitely generated ideal of R is finitely generated, the hypothesis in part (e) implies that each prime ideal of R is finitely generated, and thus R is Noetherian. □

3. Glad domains

In this section we give some results on the class of *glad* domains which were defined in [17] and give their relationship to the rings described in Theorem 2.3. By an *overring* of a domain D we mean a ring R with the same quotient field as D which contains D as a subring. Following [17] we say a domain D with quotient field $K \neq D$ is a *glad* domain if the following hold:

(1) $D = \bigcap_{i \in \Lambda} V_i$ where $\{V_i \mid i \in \Lambda\}$ is a family of Noetherian valuation overrings of R. Let v_i be the normed additive valuation associated with V_i, and let M_i be the maximal ideal of V_i.

(2) There is a monic polynomial $f \in D[X]$ of degree > 1 such that for each $i \in \Lambda$ and each $a \in V_i$, $f(a)$ is a unit of V_i.

(3) For each $a \in D$ the set $\{v_i(a) \mid i \in \Lambda\}$ is bounded.

(4) There exists $t \in D$ such that $tV_i = M_i$ for each $i \in \Lambda$.

(5) There exists a finite subset T of D which is a set of representatives for V_i/M_i for each $i \in \Lambda$.

Before considering the relationship of glad domains to the rings R for which $Int(R)_P = Int(R)_{(R-P)}$ for each maximal ideal P of R, we first give an alternate characterization of glad domains. (For simplicity of some of our later statements we are departing slightly from the definition in [17] by not requiring the family $\{V_i \mid i \in \Lambda\}$ to be infinite. Thus, we include the case that D is Dedekind.)

Proposition 3.1. *Let D be an almost Dedekind domain with quotient field $K \neq D$. Then D is a glad domain if and only if the following statements hold:*

(a) *Each principal ideal of D contains a power of its radical.*

(b) *The Jacobson radical J of D is a nonzero principal ideal.*

(c) *There exists a finite subset T of D which is a set of representatives for D/P for each maximal ideal P of D.*

Proof. If D is a glad domain then (b) and (c) hold by [17, Proposition 9]. To prove statement (a), let $a \in D$ and let $I = \mathrm{rad}(aD)$. Let $k = \sup\{v_i(a) \mid i \in \Lambda\}$, and let $b_1, \dots, b_k \in I$. Now if $i \in \Lambda$ we have $v_i(b_1 b_2 \cdots b_k) \geq v_i(a)$. Thus $(b_1 b_2 \cdots b_k)/a \in V_i$ for each i, and therefore $b_1 b_2 \cdots b_k \in aD$. Thus, $I^k \subseteq aR$.

Conversely, assume D is almost Dedekind domain satisfying (a)–(c). Then property (1) holds with $\{V_i \mid i \in \Lambda\} = \{D_P \mid P$ is a maximal ideal of $R\}$. For (2), let $T = \{a_1, \ldots, a_q\}$ be a set of representatives for each residue field of D, which exists by (c). Then $f = 1 + \prod_{i=1}^{q}(X - a_i)$ is clearly a unit valued polynomial for D_P for each maximal ideal P of D. For property (3) let $a \in D$ and let $I = \mathrm{rad}(aD)$. By property (a) we have $I^n \subseteq aR$ for some $n \geq 1$. Thus if P is a maximal ideal of D we have $P^n D_P \subseteq I^n D_P \subseteq aD_P$. Thus $v_P(a) \leq n$, where v_P denotes the normalized valuation for D_P. Thus (3) holds. Properties (4) and (5) above follow immediately from (b) and (c). □

Corollary 3.2. *In a glad domain, the radical of a finitely generated ideal is finitely generated.*

Proof. Let I be a nonzero finitely generated ideal of the glad domain R and let tR be the Jacobson radical of R. Then $\mathrm{rad}(I) = (I, t)R$. Indeed it suffices to check this locally at primes P containing I. But for such a prime P we have $\mathrm{rad}(I)R_P = PR_P = (I, t)R_P$. □

The next result shows that the glad domains play a role in the class of almost Dedekind domains R with finite residue fields for which $Int(R_S) = Int(R)_S$, which is similar to the place occupied by the Noetherian valuation rings in the class of Dedekind domains.

Proposition 3.3. *Let $R = \bigcap_{i=1}^{m} R_i$ with each R_i a glad overring of R. Then the following hold:*

(1) *$R_i = R_S$ where $S = R - \bigcup\{Q \cap R \mid Q$ a maximal ideal of $R_i\}$.*
(2) *R is an almost Dedekind domain with finite residue fields.*
(3) *R is Bezout.*

Proof. For (1) we can adapt the proof from [15, Theorem 107]. Let $R_i = \bigcap\{V_{ij} \mid j \in \Lambda_i\}$ for $i = 1, \ldots, m$, where V_{ij} is a Noetherian valuation ring with maximal ideal M_{ij}. By the definition of glad domain, for each fixed i, all of the residue fields V_{ij}/M_{ij} are isomorphic. Let q_i be the order of V_{ij}/M_{ij}, $j \in \Lambda_i$. Let $k = q_1 q_2 \cdots q_m (q_1 - 1)(q_2 - 1) \cdots (q_m - 1)$. Let $x \in R_1$. It follows that for each positive integer n which is relatively prime to k, and for each V_{ij} in which x is a unit, the element $u = 1 + x + \cdots + x^{n-1}$ is also a unit of V_{ij}. Indeed

(a) if x maps to 1 in V_{ij}/M_{ij}, this follows since n is not divisible by the characteristic of V_{ij}/M_{ij}, and

(b) if the image x^* of x in V_{ij}/M_{ij} is not 1, $u = 1 + x^* + \cdots + (x^*)^{n-1} = [1 - (x^*)^n]/(1 - x^*)$ is a unit since n is relatively prime to the order of the element x^* in the group of units of V_{ij}/M_{ij}.

Now let $P_{ij} = M_{ij} \cap R$. We must show that $R_i = R_{S_i}$, where $S_i = R - \bigcup\{P_{ij} \mid j \in \Lambda_i\}$, and we may take $i = 1$. It is clear that $R_{S_1} \subseteq R_1$. Let $x \in R_1$, and choose k so that

$u = 1+x+\cdots+x^{n-1}$ is as in the above paragraph. Let $s = 1/u$. We have the following cases:

(i) If $x \in M_{ij}$, then $s \in V_{ij} - M_{ij}$.

(ii) If $x \in V_{ij} - M_{ij}$, then $s \in V_{ij} - M_{ij}$.

(iii) If $x \notin V_{ij}$, then $y = 1/x \in M_{ij}$, and then $s = y^{n-1}/(1 + y + \cdots + y^{n-1}) \in M_{ij}$.

Thus $s \in R$. Also since $x \in R_1$, $s \in V_{1j} - M_{1j}$ for each j. Thus, $s \in R - P_{1j}$ for each j. It remains to show that $sx \in R$. That is we must show $sx \in V_{ij}$ for each i and j. Thus we need only consider case (iii) above. In this case we have

$$sx = y^{n-2}/(1 + y + \cdots + y^{n-1}) \in V_{ij}.$$

This proves (1). The statement (2) follows easily from this.

For (3) let $I \neq \{0\}$ be a finitely generated ideal of R. It follows that the Jacobson radical J of R is non-zero. Let $t \in J - \{0\}$ and $y \in I - \{0\}$. Then by [14, Theorem 3.1] there exists an element $x \in I$ such that $I = xR + ytR$. But then $I \subseteq xR + JI$, and thus $I = xR$ by Nakayama's Lemma. □

Theorem 3.4. *Let R be an almost Dedekind domain with finite residue fields. Then $Int(R_P) = Int(R)_{(R-P)}$ for each $P \in Spec(R)$ if and only if for each non-zero $a \in R$, there is a partition $\{\mathscr{T}_1, \ldots, \mathscr{T}_m\}$ of $V(aR)$ into open sets such that the ring $R_i = \bigcap\{R_Q \mid Q \in \mathscr{T}_i\}$ is a glad domain for $i = 1, \ldots, m$.*

Proof. ($\Rightarrow$) Let $a \in R$ and let $I = \mathrm{rad}(aR)$. It follows from Theorem 2.3 that we may let $I = (y_1, \ldots, y_k)R$, and that there is a partition $\{\mathscr{T}_1, \ldots, \mathscr{T}_m\}$ of $V(aR)$ into open sets such that for each $i = 1, \ldots, m$ we have:

(a) a set $T_i \subseteq R$ which is a set of representatives of R/Q for each $Q \in \mathscr{T}_i$, and

(b) an element $t_i \in \{y_1, \ldots, y_k\}$ such that $t_i R_Q = QR_Q$ for each $Q \in \mathscr{T}_i$.

Let $R_i = \bigcap\{R_Q \mid Q \in \mathscr{T}_i\}$ for $i = 1, \ldots, m$. We claim that each R_i is a glad domain. To see this we observe that the family $\{R_P \mid P \in \mathscr{T}_i\}$ is a family of valuation overrings of R_i satisfying the five properties of the definition of glad domain. Indeed these are all immediate from the definition of the $\mathscr{T}_i$ except possibly (3). For this let $b \in R_i$. Then $b = c/d$, $c, d \in R$. Let H be the radical of cR in R, let n be such that $H^n \subseteq cR$, and let $P \in \mathscr{T}_i$. Then $v_P(c) \leq n$, and from $db = c$ we get $v_P(d) + v_P(b) = v_P(c) \leq n$. Since $v_P(d) \geq 0$, we get $v_P(b) \leq n$.

($\Leftarrow$) If for each nonzero $a \in R$ there is a partition as above, then by the proof of Theorem 2.3 it follows that $Int(R_P) = Int(R)_{(R-P)}$ for each $P \in \mathrm{Spec}(R)$. (Alternately, from Proposition 3.1 it follows that R satisfies (3) of Theorem 2.4.) □

The examples of non-Noetherian rings R with $Int(R)$ Prüfer given in [16, 17] are glad domains, and the examples of such rings given in [10] are easily seen to be finite intersections of glad domains. The example [8, 6.4] is an example of an almost Dedekind domain with finite residue fields for which $Int(R)_S = Int(R_S)$ for each multiplicative subset of R which is not a finite intersection of glad domains.

4. Generating ideals in *Int*(*R*)

In this section we show that if R is an almost Dedekind domain with finite residue fields such that $Int(R_P) = Int(R)_{(R-P)}$ for all maximal ideals P of R, then each finitely generated ideal of $Int(R)$ can be generated by two elements, and if R is a finite intersection of glad overrings, then one of the generators may be chosen arbitrarily. This result extends results on the case that R is Noetherian in [21, 7, 18] and includes all known examples where $Int(R)$ is Prüfer. Recall that since $Int(R)$ has Krull dimension two in this case, a theorem of Heitmann [14, Theorem 3.1] insures that each finitely generated ideal of $Int(R)$ is generated by three elements. We recall some results and definitions from [18].

Definition 4.1. An ideal $\mathscr{I}$ of $Int(R)$ is said to be *unitary* if $\mathscr{I} \cap R \neq \{0\}$. Let $\mathscr{I}(a) = \{f(a) \mid f \in \mathscr{I}\}$. The domain $Int(R)$ is said to have the *strong Hilbert property* if for finitely generated unitary ideals $\mathscr{I}$ and $\mathscr{J}$ of $Int(R)$, $\mathscr{I}(a) = \mathscr{J}(a)$ for each $a \in R \Rightarrow \mathscr{I} = \mathscr{J}$.

Lemma 4.2 (McQuillan [18, Lemma 2.6]). *If R is an integral domain such that $Int(R)$ is a Prüfer domain then $Int(R)$ has the strong Hilbert property.*

Lemma 4.3 (McQuillan [18, Lemma 3.2]). *Let R be an integral domain and $\mathscr{I}$ a finitely generated unitary ideal of $Int(R)$. Then there is a nonzero ideal J of R such that $a, b \in R$, and $a - b \in J \Rightarrow \mathscr{I}(a) = \mathscr{I}(b)$.*

An ideal J as in the above lemma is called a *period* for $\mathscr{I}$.

Theorem 4.4. *Let R be an almost Dedekind domain with finite residue fields such that $Int(R_P) = Int(R)_{(R-P)}$ for each maximal ideal P of R. Then each finitely generated ideal $\mathscr{I}$ of $Int(R)$ can be generated by two elements. If $\mathscr{I}$ is unitary, then any nonzero element of $\mathscr{I} \cap R$ may be chosen as one of the two generators of $\mathscr{I}$.*

Proof. Let K be the quotient field of R. If $\mathscr{I}$ is not unitary, let $S = R - \{0\}$, and let A be a finite generating set for $\mathscr{I}$. Then $\mathscr{I}(Int(R)_S) = A(Int(R_S)) = fK[X]$ for some $f \in \mathscr{I}$. Then $A = fA_1$ for some finite subset A_1 of $K[X]$. Let $r \in S$ be such that $rA_1 \subseteq R[X]$. Then $\mathscr{I} \cong r\mathscr{I} = rfA_1(Int(R)) \cong rA_1(Int(R))$ and $rA_1(Int(R)) = \mathscr{I}_1$ satisfies $\mathscr{I}_1 \cap R \neq \{0\}$. Thus, it suffices to show the second statement.

Let $a \in \mathscr{I} \cap R - \{0\}$ and let $J = \mathrm{rad}(aR)$. By Theorem 2.3, J is finitely generated. Then $J = (a^2, t)$ for some $t \in J$ by [14, Theorem 3.1]. It follows that $tR_Q = QR_Q$ for each $Q \in V(aR)$.

By Lemma 2.2 we can define a relation $\sim$ on the set $V(aR)$ by $P \sim Q$ if there is a subset $T \subseteq R$ such that T is a set of representatives for both R/P and R/Q. By Theorem 2.3 there are only finitely many equivalence classes $\mathscr{T}_1, \ldots, \mathscr{T}_m$ of $\sim$. For each i let T_i be a finite subset of R such that for each $Q \in \mathscr{T}_i$, T_i is a set of representatives of R/Q.

Then it follows from Lemma 2.1 that each $\mathscr{T}_i$ is open in the Zariski topology on $V(aR)$, and thus is also closed. Therefore, we can write $R/aR = (R/aR)e_1 \oplus \cdots \oplus (R/aR)e_m$, for some idempotents e_i of R/aR, where the image of e_i in $(R/aR)_P$ is the identity if $P \in \mathscr{T}_i$, and the image of e_i in $(R/aR)_P$ is zero if $P \notin \mathscr{T}_i$. Let $d_i \in R$ be a preimage of e_i for each i.

Suppose we have shown that for each i there exists $f_i \in Int(R)$ such that $\mathscr{A}_i = (a, f_i)Int(R)$ satisfies $\mathscr{I}_P = (\mathscr{A}_i)_P$ for each $P \in \mathscr{T}_i$. Let $f = d_1 f_1 + \cdots + d_m f_m$, and $\mathscr{A} = (a, f)Int(R)$. Then for $P \in V(aR)$ we have $P \in \mathscr{T}_i$ for a unique i; say $i = 1$. Then $(\mathscr{A}(x))R_P = (a, f(x))R_P = (a, f_1(x))R_P = (\mathscr{I}(x))R_P$. Also, if $a \notin P$ then $\mathscr{I}(x)R_P = R_P = \mathscr{A}(x)R_P$. Thus by Lemma 4.2, $\mathscr{A} = \mathscr{I}$. Therefore, it suffices to show that for each i there exists $f_i \in Int(R)$ such that $\mathscr{A}_i = (a, f_i)Int(R)$ satisfies $\mathscr{I}_P = (\mathscr{A}_i)_P$ for each $P \in \mathscr{T}_i$.

Let us fix j and assume that $T_j = \{a_0, a_1, a_2, \ldots, a_{q-1}\}$ is a set of representatives for R/Q for each $Q \in \mathscr{T}_j$. We also have $tR_Q = QR_Q$ for each $Q \in V(aR)$.

By Lemma 4.3 we may let $rR \subseteq aR$ be a period of $\mathscr{I}$, so that $\mathscr{I}(x) = \mathscr{I}(y)$ for x, $y \in R$ with $x - y \in rR$. Let $H = \mathrm{rad}(rR)$. Then H is finitely generated by Theorem 2.3 and thus $H^k \subseteq rR$ for some $k \geq 1$. For each $n \in \mathbb{Z}_+$ write n in its q-adic expansion

$$n = u_0 + u_1 q + u_2 q^2 + \cdots + u_r q^r, \quad 0 \leq u_i < q,$$

and define

$$s_n = a_{u_0} + a_{u_1} t + a_{u_2} t^2 + \cdots + a_{u_r} t^r.$$

Consider the ideals $\mathscr{I}(s_i) = B_i$, $i = 0, 1, \ldots, q^k - 1$. Since $\mathscr{I}$ is finitely generated, the B_i are finitely generated also. By [14, Theorem 3.1] there exist $b_i \in B_i$ such that $B_i = (a, b_i)R$. Let $N = q^k - 1$ and define F_0, F_1, $\ldots$, $F_N \in K[X]$ by

$$F_j = (X - s_{j+1})(X - s_{j+2}) \cdots (X - s_{j+N}).$$

Let $s_0! = 1$ and for $n \geq 1$ let $s_n! = \prod_{i=1}^n s_i$. Let

$$G_n = \frac{F_n}{s_N!}.$$

Since $QR_Q = tR_Q$ and T_j is a set of representatives of R/Q for each $Q \in \mathscr{T}_j$, then by [18, Corollary 4.4], for each $Q \in \mathscr{T}_j$ the polynomials $G_n(X)$ are contained in $Int(R_Q)$ and have the property that

$$G_n(x) \text{ is a unit of } R_Q \text{ if and only if } x \equiv s_n \pmod{Q^N R_Q}.$$

Let e be an exponent of the group $(R/Q^k)^*$ for each $Q \in V(aR)$, which is possible by Corollary 2.5(a). Define $f(X) \in Int(R)$ by

$$f(X) = \sum_{m=0}^{N} b_m G_m(X)^{ek}.$$

Then the polynomial $f(X)$ has the property that for any $Q \in \mathscr{T}_j$, we have

$$f(x) \equiv b_n \,(\operatorname{mod} Q^k R_Q) \text{ if } x \equiv s_n \,(\operatorname{mod} Q^N R_Q).$$

Indeed let $x \equiv s_n \,(\operatorname{mod} Q^N R_Q)$. Then $G_n(x)$ is a unit of R_Q for each $Q \in \mathscr{T}_j$. Therefore, the image $\overline{G_n(x)} \in R/Q^k$ is a unit for each $Q \in \mathscr{T}_j$. Thus, $G_n(x)^{ek} - 1 \in Q^k R_Q$ for each $Q \in \mathscr{T}_j$. If $i \in \{0, 1, \ldots, N\} - \{n\}$ then $G_i(x) \in QR_Q$ for each $Q \in \mathscr{T}_j$. Thus, $G_i(x)^{ek} \in Q^k R_Q$ for each $Q \in \mathscr{T}_j$. Thus, f has the desired property.

Let $\mathscr{A} = (a, f)(Int(R))$. To show $\mathscr{I} R_P = \mathscr{A} R_P$ for $P \in \mathscr{T}_j$ it suffices to show that $\mathscr{I}(x)R_P = \mathscr{A}(x)R_P$ for each $x \in R$. To show that $\mathscr{I}(x)R_P = \mathscr{A}(x)R_P$ for each $x \in R$, it suffices to show $\mathscr{I}(s_n)R_P = \mathscr{A}(s_n)R_P$ for $n = 1, 2, \ldots, k$. Since $a \in P$ we have $tR_P = JR_P = PR_P$, and thus $\mathscr{I}(s_n)R_P = (a, b_n)R_P = \mathscr{A}(s_n)R_P$ by the choice of f.

Therefore, $\mathscr{I}(x) = \mathscr{A}(x)$ for each $x \in R$, and therefore $\mathscr{I} = \mathscr{A}$ by Lemma 4.2. □

Corollary 4.5. *Let R be an almost Dedekind domain with finite residue fields such that $Int(R_P) = Int(R)_{(R-P)}$ for each maximal ideal P of R. Then $\mathscr{I} \oplus \mathscr{J} \cong \mathscr{I}\mathscr{J} \oplus Int(R)$ for each pair of finitely generated ideals $\mathscr{I}$ and $\mathscr{J}$ of $Int(R)$.*

Proof. The argument in [2, p. 144] shows that this follows from Theorem 4.4. □

The following is another example of where glad domains behave similarly to Noetherian valuation rings. In the Noetherian case the following result was given in [7, Theorem 7.5]. We say an ideal I of a ring A is *strongly 2-generated* if for each $a \in I - \{0\}$ there exists $b \in I$ such that $I = (a, b)A$.

Theorem 4.6. *Let R be an almost Dedekind domain with finite residue fields which is a finite intersection of glad domains. Then each finitely generated ideal $\mathscr{I}$ of $Int(R)$ is strongly 2-generated.*

Proof. As in the proof of Theorem 4.4 we may assume $\mathscr{I}$ is unitary. Let J be the Jacobson radical of R. Let $a \in (\mathscr{I} \cap R)J - \{0\}$ and let $g \in \mathscr{I} - \{0\}$. By Theorem 4.4 we may choose $f \in \mathscr{I}$ such that $\mathscr{I} = (a, f)Int(R)$. For each $b \in R$ the polynomial $h = f + ab$ has the same property as f. That is $(a, f)(Int(R)) = (a, f + ab)(Int(R)) = (a, h)(Int(R))$. Since R is not a field, R is infinite, and thus we may choose b so that $(g, h)K[X] = K[X]$.

To show that $\mathscr{I} = (g, h)Int(R)$ let $pg + qh = 1$, $p, q \in K[X]$. Then for some $c \in R$ we have $cp, cq \in R[X]$, and then $(cp)g + (cq)h = c \in \mathscr{I}$. We have $\mathscr{I} = (a, h)(Int(R)) = (c, a, h)(Int(R)) \subseteq (g, a, h)(Int(R)) \subseteq \mathscr{I}$. Thus, $\mathscr{I} = (g, a, h)(Int(R))$. To show that $\mathscr{I} = (g, h)(Int(R))$ it suffices by Lemma 4.2 to show $\mathscr{I}(x) = (g(x), h(x))R$ for each $x \in R$. But $\mathscr{I}(x) = (g(x), h(x), a)R$ and $a \in \mathscr{I}(x)J$. Thus, $\mathscr{I}(x) = (g(x), h(x))R$ by Nakayama's lemma. □

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